

The Scale Axis Picture Show

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ABSTRACT

We demonstrate how the scale axis transform can be used to compute a parameterized family of shape skeletons. The skeletons gradually represent only the most important features of a shape, in a scale-adaptive manner. Here a shape \mathcal{O} is any bounded open subset of the plane \mathbb{R}^2 . The scale axis for scale value s is the medial axis of the multiplicatively grown shape \mathcal{O}_s , where \mathcal{O}_s is the union of medial balls of \mathcal{O} with radii scaled by the factor s .

We present a simple algorithm to compute a parameterized family of skeletons for shapes that are finite unions of balls in the plane. The algorithm is based on the scale axis transform. We compare the computed family of skeletons with two medial axis filters, namely the λ -medial axis, and a filter based on an angle criterion.

Categories and Subject Descriptors

F.2.2 [Nonnumerical Algorithms and Problems]: Geometrical problems and computations; I.3.5 [Computational Geometry and Object Modeling]: Curve, surface, solid, and object representations

General Terms

Theory, Algorithms

Keywords

medial axis, union of balls, skeleton

1. INTRODUCTION

One of the most popular shape skeletons is the medial axis. Here a shape \mathcal{O} is an open subset of the Euclidean space. The medial axis transform $\text{MAT}(\mathcal{O})$ is the set of all maximal, empty, and open balls. A ball $B(c, r) \subseteq \mathcal{O}$ centered at $c \in \mathcal{O}$ with radius r is called maximal, if it is not contained in any other open ball contained in \mathcal{O} . A ball in the medial axis transform is called a medial ball, and the union of the centers of all medial balls is called the medial axis of the

shape \mathcal{O} . Note that any medial ball touches the boundary of \mathcal{O} at least in two points.

Despite its simple and elegant definition, the medial axis is plagued with two major problems:

- (1) The medial axis can only be computed exactly for very restricted classes of shapes, among them finite unions of balls [3, 6].
- (2) The medial axis is not stable under small perturbations of a shape, i.e., perturbing a shape slightly (for example in terms of Hausdorff distance) can result in substantially different medial axes.

The first problem can be mitigated by the fact that any shape can be approximated by a finite union of balls [2]—in a sense by a discretization of the medial axis transform. The second problem is more challenging and has been addressed by several researchers in recent years. One paradigm that emerged as the dominant strategy to compute a stable skeleton is to filter the medial axis, i.e., to assign a score to every point of the medial axis and to retain only those medial axis points that have score larger (or smaller, respectively) than a prescribed threshold. Examples of the filtering paradigm are the λ -medial axis [4] and angle based methods that consider the angle formed by the medial axis and the points on the boundary of the shape touched by the corresponding medial axis ball [5, 1, 9]. For the λ -medial axis there is even a proof of stability. Nevertheless, so far the filtering approaches induce yet another problem, namely the medial axis is not pruned in a feature adaptive way. That is, the filtered medial axis though stable may no longer faithfully represent the features of the shape. To address this problem we introduced the scale axis transform [7] which provides a parameterized family of skeletons (that are not necessarily subsets of the medial axis). At the moment we are lacking a proof of stability for the scale axes, but in experiments (some of which we show in the video) it behaves stable while still faithfully representing the features of the shape for a reasonable interval of the scale parameter values. Like the medial axis, the scale axes cannot be computed exactly for most shapes, but fortunately it can be approximated for finite unions of balls. In the following we give some more details on the definition of the scale axis, its approximation for a finite union of balls and compare it to the aforementioned medial axis filters.

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2. THE SCALE AXIS TRANSFORM

The definition of the scale axis transform [7] is based on the multiplicative scaling operation that is designed to eliminate “locally small” features. Using the medial axis transform, a shape can always be considered as a union of balls, where every ball contributes to the description of the shape. Therefore, the task of finding locally small features can be posed as the problem of finding locally small balls, i.e. balls that have a significantly larger ball close to them. We detect such configurations using the multiplicative scaling, operation that grows the radius of every medial ball by a factor $s > 1$:

DEFINITION 1 (MULTIPLICATIVE SCALING). *For an open set \mathcal{O} and $s > 0$, the multiplicatively s -scaled shape is $\mathcal{O}_s = \bigcup_{B(c,r) \in \text{MAT}(\mathcal{O})} B(c, sr)$.*

Since growing is only used for detecting this contrast in ball sizes, we compensate for the overall growth of the shape and define the scale axis transform as the set of scaled-back medial balls of the grown shape.

DEFINITION 2 (SCALE AXIS TRANSFORM). *For $s \geq 1$, the s -scale axis transform of an open set $\mathcal{O} \subset \mathbb{R}^d$ is $\text{SAT}_s(\mathcal{O}) = \{B(c, r/s) \mid B(c, r) \in \text{MAT}(\mathcal{O}_s)\}$. We call the set of centers of the balls in $\text{SAT}_s(\mathcal{O})$ the s -scale axis.*

3. THE ALGORITHM

Computing the scale axis transform for general shapes is not feasible since the construction relies on manipulations of an infinite number of balls and the computation of medial axis transform. Therefore, we designed a simple algorithm that uses the concept the scale axis transform to compute a family of scale-aware skeletons.

The input to our algorithm is a union of a finite number of balls in the plane. This class of shapes has a particularly simple medial axis structure composed of segments and points. We can compute the exact medial axis of such a shape with the algorithm of Amenta and Kolluri [3] or special unions by using the slightly more efficient algorithm [6].

Our algorithm to compute skeletal structures follows the construction of the scale axis transform. Instead of scaling an infinite number of medial balls we work with a finite sampling of the medial balls. The sampling of the medial axis is done in a recursive manner. For every segment of the medial axis we pick as samples the endpoints of the segment. If the two corresponding medial balls are not intersecting in an angle larger than a threshold (178° in our implementation), then we choose the midpoint of the segment as a sample and check the two new segments for intersection depth and recurse if needed.

The steps to compute the s -skeleton are the following:

1. Compute the medial axis of the input U (union of balls)
2. Sample the medial axis of U
3. Grow the medial balls corresponding to the samples by multiplying their radii by s
4. Compute the medial axis of the grown shape, i.e., the union of the grown balls
5. Regularize the medial axis (see details below)
6. Sample the medial axis of the grown shape
7. Shrink the medial balls corresponding to the new samples with factor s

The regularization step is needed in order to compensate for discretization artifacts. Small hairs can appear in the medial axis of the grown shape in case that the input medial balls have contact arcs on the boundary of the shape — many medial balls of the union of a finite set of balls have this property. This regularization step simply repeats the above “grow balls–medial axis computation–shrink balls” steps, this time for the grown shape, with a tiny growth factor, i.e., 1.01 in our implementation.

4. COMPARISON

As mentioned above, the most popular approach to compute stable skeletal structures is to filter the medial axis. To decide whether a medial axis point has to be pruned, the following two measures are very often used: the radius of the minimal enclosing ball of contact points — λ -medial axis [4], and the angle formed by the contact points and the medial axis point [5, 1, 9].

To implement medial axis filtrations based on the above measures we start with a dense sampling of the boundary. From the samples we can compute the medial axis of inner Voronoi balls as a subset of the Voronoi diagram [6]. We filter this set of segments by computing either the radius of the minimal enclosing ball of contact points or the angle formed by the contact points with the midpoint of the segment.

All the algorithms were implemented in an extended version of Mesecina [8], the software that has been used to generate the images and animations presented in the video.

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